Covariance Analysis in SISO Linear Systems Identification

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Abstract

In this paper we analyse the asymptotic covariance of models of causal single-input single-output linear time invariant systems. Expressions for the asymptotic (co)variance of system properties estimated using the prediction error method are derived. These expressions delineate the impacts of model structure, model order, true system dynamics, and experimental conditions. A connection to results on frequency function estimation is established. Also, simple model structure independent upper bounds are derived. Explicit variance expressions and bounds are provided for common system properties such as impulse response coefficients and non-minimum phase zeros. As an illustration of the insights the expressions provide, they are used to derive conditions on the input spectrum which make the asymptotic variance of non-minimum phase zero estimates independent of the model order and model structure.

1 Introduction

In system identification, as in all types of modeling, it is important to be able to assess the model error. Different assumptions on the system and the noise lead to different ways to quantify the model error, see Ninness and Goodwin (1995, Chapter 5). Assuming the noise to be stochastic and that the system can be described by a model within the used model set leads to error quantification using confidence ellipsoids based on the asymptotic covariance matrix of the parameter estimates (Ljung; 1999). Also techniques for non-asymptotic confidence regions have been developed (Campi and Weyer; 2005, 2010; Douma and Van den Hof; 2006; Csáji et al.; 2012b,a; Kolumbán et al.; 2015). In this contribution, though, we will focus on the traditional asymptotic covariance matrix which in many cases give reliable information of the model error (Garatti et al.; 2004).

We will consider prediction error identification of causal single-input single-output (SISO) finite dimensional linear time invariant (LTI) systems. The unknown system parameters will be denoted by $\theta = [\theta_1, \cdots, \theta_n]^{\mathrm{T}} \in \mathbb{R}^n$, with θ^o denoting the true value and we will assume that the true system is in the model class. We will assume that (see (Ljung; 1999) for exact conditions) the parameter estimate $\hat{\theta}_N \in \mathbb{R}^n$ has the property that the (normalized) model error $\sqrt{N}(\hat{\theta}_N - \theta^o)$ becomes normally distributed as the sample size N of the data set grows to infinity:

$$\sqrt{N}\left(\hat{\theta}_N - \theta^o\right) \in \mathrm{As}\mathcal{N}\left(0, \mathrm{AsCov}\,\hat{\theta}_N\right). \tag{1}$$

The asymptotic covariance matrix $\mathrm{AsCov}\,\hat{\theta}_N$ of the limit distribution is a measure of the model accuracy. This is reinforced by that, under mild additional conditions (Ljung; 1999),

$$\lim_{N\to\infty} N \cdot \mathbf{E}\left[(\hat{\theta}_N - \mathbf{E}\hat{\theta}_N)(\hat{\theta}_N - \mathbf{E}\hat{\theta}_N)^{\mathrm{T}}\right] = \mathrm{AsCov}\,\hat{\theta}_N. \tag{2}$$

Under the assumptions above

$$\mathrm{AsCov}\,\hat{\theta}_N = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi(\mathrm{e}^{\mathrm{j}\omega})\Psi^*(\mathrm{e}^{\mathrm{j}\omega})\mathrm{d}\omega\right]^{-1}, \tag{3}$$

where Ψ is the gradient of the one-step ahead output predictor and where superscript * denotes complex conjugate transpose. We will use $\langle \Psi, \Psi \rangle$ to denote the integral on the right-hand side of (3) in the following. However, our interest will not primarily be the model parameters θ themselves but some "system theoretic" quantity. We will let such a quantity be represented by a differentiable function $J : \mathbb{R}^n \to \mathbb{C}^{1\times p}$. Given an estimate $\hat{\theta}_N$ of θ^o, a natural estimate of $J(\theta^o)$ is $J(\hat{\theta}_N)$ with asymptotic co-

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variance [1]

$$\operatorname{AsCov} J(\hat{\theta}_N) \coloneqq \lim_{N\to\infty} N \cdot \mathbf{E}\left[(J(\hat{\theta}_N) - J(\theta^o))^*(J(\hat{\theta}_N) - J(\theta^o))\right].$$

Using a Taylor series expansion around $J(\theta^o)$ and (3), it can be shown (Ljung; 1999), that

$$\operatorname{AsCov} J(\hat{\theta}_N) = \Lambda^* \left[\langle \Psi, \Psi \rangle\right]^{-1} \Lambda,$$

where

$$\Lambda \coloneqq J'(\theta^o) \in \mathbb{C}^{n\times p}. \tag{4}$$

We shall be slightly more general and allow for $\langle \Psi, \Psi \rangle$ to be singular, in which case

$$\operatorname{AsCov} J(\hat{\theta}_N) = \Lambda^* \left[\langle \Psi, \Psi \rangle\right]^{\dagger} \Lambda. \tag{5}$$

The motivation for using the Moore-Penrose pseudo-inverse $[\langle \Psi, \Psi \rangle]^{\dagger}$ stems from that this gives the correct covariance for properties that are uniquely defined by the data even if the parameter estimate is non-unique, see Mårtensson and Hjalmarsson (2011, Section I) and Hjalmarsson (2009); Stoica and Marzetta (2001) for details.

Our main assumption is that prediction error identification results in an asymptotic covariance AsCov $J(\hat{\theta}_N)$ of the quantity of interest J given by (5). We refer to Ljung (1999) for exact conditions for when this holds.

When the model structure, the true system and the experimental conditions are known, it is straightforward to compute (5) numerically. However, such a procedure typically reveals little in terms of how system properties and design variables (model order, model structure, experimental conditions etc.), influence the asymptotic covariance. In Mårtensson and Hjalmarsson (2011) (see also Mårtensson and Hjalmarsson (2009)), a geometric approach is used to re-express (5) in a form more tangible for interpretation. The use of the technique is illustrated by analyzing the impact that system complexity, additional inputs and additional sensors have on the asymptotic covariance. Our work is based on this idea and we will derive expressions for (5) for a class of system properties such as frequency response, impulse response coefficients, poles and zeros, and system norms. This technique has been applied to various settings, e.g., cascade systems (Everitt et al.; 2013), single-input multi-output systems (Ramazi et al.; 2014), error-in-variables identification (Hjalmarsson et al.; 2011), identification for minimum variance control (Mårtensson et al.; 2011).

[1] This definition is slightly non-standard in that the second factor is usually conjugated. For the standard definition, all results in the paper have to be transposed.

A case that has attracted significant interest in the past is the variance of frequency function estimates $G(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N)$. For the prediction error method it was shown in Ljung (1985) that

$$\lim_{m\to\infty} \frac{1}{m} \operatorname{AsCov} G(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N) = \frac{\Phi_v(\mathrm{e}^{\mathrm{j}\omega})}{\Phi_u(\mathrm{e}^{\mathrm{j}\omega})}, \tag{6}$$

where m is the model order and Φ_u and Φ_v are the spectral densities of the input signal and noise, respectively. This simple and elegant expression, which is valid for open loop identification, revealed that for large model orders, the accuracy of the frequency function estimate does not depend on the model structure or the number of estimated parameters, but only the model order m (which may be different from the number of estimated parameters). The results suggests that the covariance grows linearly with the model order. Furthermore, it shows that the accuracy of the frequency function estimate at a particular frequency only depends on the input and noise spectrum at that particular frequency. Various refinements can be found in Hildebrand and Gevers (2004); Hjalmarsson and Ninness (2006); Ninness and Hjalmarsson (2004); Xie and Ljung (2001, 2004); Wahlberg et al. (2012).

The frequency function result in Ljung (1985) also covered closed loop identification using input and output measurements as data, and was extended to some alternative closed loop identification methods in Gevers et al. (2001). Covariance expressions for finite model orders were presented in Ninness and Hjalmarsson (2005). The asymptotic covariance of the parameter estimates for Box-Jenkins models were studied in Forssell and Ljung (1999) for a range of different closed loop identification methods. Additional contributions to quantify frequency response errors are Bombois et al. (2005); Campi and Weyer (2005); Schoukens et al. (2006).

Parallel to the interest in the accuracy of frequency response estimates, there has been a series of results regarding the accuracy of estimated non-minimum phase (NMP) zeros and unstable poles, the interest arising due to the importance of such zeros and poles in control. For poles and zeros of amplitude larger than one, the main conclusion is that the asymptotic variance approaches a finite limit as the model order tends to infinity (Lindqvist; 2001; Mårtensson and Hjalmarsson; 2009). This is remarkable compared to the linear growth for frequency function estimates. A related and very interesting contribution is the paper Gevers et al. (2009) where conditions for the minimum degree of richness of the excitation required for the information matrix $\langle \Psi, \Psi \rangle$ to be non-singular are established.

Promising methods based on hypothesis testing are also emerging, which, under very mild assumptions on the noise distribution, provide non-asymptotic confidence

regions (Csáji et al.; 2012b,a; Kolumbán et al.; 2015). In Csáji et al. (2012a,b), the noise terms are assumed independent and symmetrically distributed around zero, and in Kolumbán et al. (2015) it is shown that the symmetry requirement may be replaced by exchangeability.

Contributions and outline

As pointed out in Mårtensson and Hjalmarsson (2011), the geometric approach has its origin in Ninness and Hjalmarsson (2004), where exact expressions for the asymptotic variance of frequency function estimates for LTI models were derived using the theory of reproducing kernels, a theory which is based on orthogonal projections. Our contribution to the characterization of the variance error for estimates of LTI systems can be seen as an extension of that work to general system properties J, and thus extends Mårtensson and Hjalmarsson (2011), using new techniques which deepens the geometrical interpretation of (5). As a result, our contribution provides an alternative system theoretic interpretation of the results in Ninness and Hjalmarsson (2004), see Section 6.1, thus furthering the understanding of frequency function estimation.

More precisely, the contributions of this paper are:

i) Section 3.2: *Re-parametrization formulae.* We use an orthonormal-basis-parametrization of the system model and the quantity of interest in order to re-express (5) (Lemma 2 and Lemma 3).
ii) Sections 4 and 5: *A general characterization of (5) for Linear Time Invariant systems.* Here we provide general formulas for (5), valid for different experimental conditions and model parametrizations, based on the re-parametrization mentioned above (Theorem 1 and Theorem 4).
iii) *Model structure independent upper bounds for (5).* At present there are surprisingly few rules of thumbs available regarding model quality in system identification; the expression (6) for the frequency function estimate and some similar variance expressions for pole/zero estimates are sole exceptions. Thus, determining a suitable experiment length and excitation in order to achieve a certain accuracy of, for example, an impulse response coefficient or an estimate of the $\mathcal{L}_2$ gain of the system, requires extensive calculations based on (5). A spin-off of our new expression for (5) is that it is easy to provide simple model structure independent upper bounds for (5) (Theorem 4). We hope this to be of value to practitioners.
iv) Section 6: *Expressions for the asymptotic covariance for some properties of LTI systems.* We provide expressions and upper bounds for the asymptotic variance of estimated frequency functions, impulse response coefficients, $\mathcal{L}_2$-gains and NMP-zeros.

2 Technical preliminaries

This section introduces the notation and presents the mathematical tools that are necessary for deriving and understanding the results of this paper.

2.1 Notation

The conjugate transpose of a complex-valued matrix $Z \in \mathbb{C}^{n\times m}$ is denoted Z^*. For invertible matrices, $Z^{-*} := (Z^{-1})^*$. Let $\mathbb{D}$ denote the unit disc: $\{z : |z| < 1\}$, $\mathbb{E}$ the exterior of the unit disc including infinity: $\{z : |z| > 1\}$, and $\mathbb{T}$ the unit circle: $\{z : |z| = 1\}$. For functions $f : \mathbb{C} \to \mathbb{C}^{n\times m}$, $f^*(z) := (f(z))^*$ denotes the conjugate transpose of $f(z)$. The matrix inverse of $f(z)$, if it exists, is denoted $f^{-1}(z) := (f(z))^{-1}$. Similarly, $f^{-*}(z) := (f(z))^{-*}$. In what follows, we will mainly be concerned with functions $f : \mathbb{T} \to \mathbb{C}^{n\times m}$ that arise as the restrictions of $f : \mathbb{C} \to \mathbb{C}^{n\times m}$ to the domain $z = \mathrm{e}^{\mathrm{j}\omega}, \omega \in [0, 2\pi]$. This paper will alternate between the notation $f(\omega)$, $f(\mathrm{e}^{\mathrm{j}\omega})$, and $f(z)$ as is convenient. We will consider vector valued complex functions as row vectors and the inner product of two such functions $f, g : \mathbb{T} \to \mathbb{C}^{1\times m}$ is defined as $\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)g^*(\omega)\mathrm{d}\omega$. When f and g are matrix-valued functions, we will still use the notation $\langle f, g\rangle$ to denote $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega)g^*(\omega)\mathrm{d}\omega$ whenever the dimensions of f and g are compatible. In particular, $\langle f, f\rangle$ is the Gramian matrix of the rows of f. When $W : \mathbb{T} \to \mathbb{C}^{m\times m}$ is such that $W(z)$ is a positive definite hermitian matrix for all z, the $\mathcal{L}_2^W$-norm of $f : \mathbb{T} \to \mathbb{C}^{n\times m}$ is given by $\|f\|_W = \sqrt{\mathrm{Tr}\,\langle fW, f\rangle}$ where Tr denotes the Trace operator. When $W \equiv I$ (the identity), we write $\|f\|$ and denote this the $\mathcal{L}_2$-norm of f. The space $\mathcal{L}_2^{n\times m}$ consists of all functions $f : \mathbb{T} \to \mathbb{C}^{n\times m}$, such that $\|f\| < \infty$. When $n = 1$, the notation is simplified to $\mathcal{L}_2^m$. For $f : \mathbb{T} \to \mathbb{C}^{n\times m}$, $f_i : \mathbb{T} \to \mathbb{C}^{1\times m}$ denotes the ith row of f. $\mathcal{H}_2^m$ is defined as the $\mathcal{L}_2^m$-functions that are analytic in $\mathbb{E}$. If $\Psi \in \mathcal{L}_2^{n\times m}$ for some positive integers n and m, then $\mathcal{S}_\Psi$ denotes the span of the rows of Ψ. For $f \in {\mathcal{L}_2}^{p\times m}$ and $\mathcal{S} \subseteq \mathcal{L}_2^m$, $\mathbf{P}_{\mathcal{S}}\{f\}$ denotes the orthogonal projection of f onto S, meaning that the rows of $\mathbf{P}_{\mathcal{S}}\{f\}$ are the corresponding rows of f projected on $\mathcal{S}$. For a differentiable function $f : \mathbb{R}^n \to \mathbb{C}^{1\times p}$, $f'(x_o)$ is a $n \times p$ matrix with $\frac{\partial f_j(x)}{\partial x_i}\big|_{x=x_o}$ as ijth entry, the partial derivative $\frac{\partial f(\bar{x})}{\partial x_i}$ is defined analogously. $A^\dagger$ denotes the Moore-Penrose pseudo-inverse of A.

2.2 Geometric tools for variance analysis

Our results are based on Hjalmarsson and Mårtensson (2011), which presents an expression for the asymptotic covariance (5). The main result is restated here for completeness.

Lemma 1 *Let $\Psi \in \mathcal{L}_2^{n\times m}$ and $\Lambda \in \mathbb{C}^{n\times p}$. Suppose that*

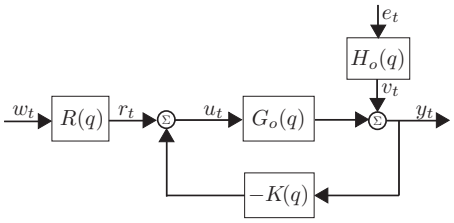

Figure 1. Block diagram of SISO LTI system with output feedback

$\gamma \in \mathcal{L}_2^{p\times m}$ *is such that*

$$\Lambda = \langle \Psi, \gamma \rangle. \tag{7}$$

Then

$$\Lambda^* \langle \Psi, \Psi \rangle^\dagger \Lambda = \langle \mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\}, \mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\} \rangle \,, \tag{8}$$

where $\mathcal{S}_\Psi$ *is the row span of* Ψ*.*

Proof The proof can be found in Hjalmarsson and Mårtensson (2011, Theorem II.5). □

With Λ as in (4), Lemma 1 gives an expression for the asymptotic covariance (5):

$$\text{AsCov}\, J(\hat{\theta}_N) = \langle \mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\}, \mathbf{P}_{\mathcal{S}_\Psi}\{\gamma\} \rangle \,. \tag{9}$$

There are many functions γ for which (7) holds, so there is a large degree of freedom in the choice of γ when analyzing the expression (9). In Hjalmarsson and Mårtensson (2011, Lemma II.8) it is shown that all solutions $\gamma \in \mathcal{L}_2^{p\times m}$ to the equation $\Lambda = \langle \Psi, \gamma \rangle$ are given by

$$\gamma = \Lambda^* \langle \Psi, \Psi \rangle^\dagger \Psi + s^\perp, \tag{10}$$

where $s^\perp$ is any $\mathcal{L}_2^{p\times m}$-function orthogonal to $\mathcal{S}_\Psi$. We will explore this degree of freedom in the next section, where a re-parametrization of $J(\theta)$ is used to find an expression for a γ that fulfills the condition (7) when $\Lambda = J'(\theta_o)$, cf. (4).

3 SISO LTI systems

In this section we present the system and model assumptions, and provide the reparametrization formulae that can be used to characterize the covariance of specific system properties.

3.1 System and model assumptions

Throughout the paper we will assume that the true system is given by a causal finite dimensional SISO LTI system $G_o(\mathrm{q})$ (q is the forward shift operator) as depicted in Figure 1, where u_t and y_t represent the measured input and output, respectively, and where e_t and w_t are uncorrelated zero mean white noise sequences with variances λ_o and 1, respectively. The causal finite dimensional LTI filter R represents a stable minimum phase spectral factor of the reference signal r_t, and H_o is an inversely stable finite dimensional LTI filter that is normalized to be monic, i.e., $\lim_{z\to\infty} H_o(z) = 1$. The system G_o includes at least one unit time delay, so that the feedback loop is well defined, and we also assume the entire system to be internally stabilized by the causal finite dimensional LTI controller K. Furthermore, we will assume that neither G_o nor K have poles on the unit circle. The system is said to be operating in open loop when $K = 0$. Next, we introduce a quite general family of model structures that will be covered.

The system is modeled by

$$y_t = T(\mathrm{q}, \theta)\chi_t, \qquad \chi_t = [u_t,\ e_t]^{\mathrm{T}}, \tag{11}$$

where $T(\mathrm{q}, \theta) = [G(\mathrm{q}, \theta), H(\mathrm{q}, \theta)]$ is a causal finite dimensional LTI model of the system and the noise dynamics, parameterized by the vector $\theta \in \mathbb{R}^n$. It is assumed that the model parametrization is such that the *true* system is in the model set, that is, there is a, not necessarily unique, parameter θ^o such that

$$G_o(\mathrm{q}) = G(\mathrm{q}, \theta^o), \quad H_o(\mathrm{q}) = H(\mathrm{q}, \theta^o). \tag{12}$$

The model $T(z, \theta)$ is continuously differentiable with respect to θ in a neighborhood of θ^o. The type of model described above includes all standard black-box model structures such as ARMAX, output error and Box-Jenkins.

Now, let Φ_v and Φ_χ denote the spectrum of v_t and χ_t, respectively, and introduce the spectrum of the signal-to-noise ratio

$$\Phi_{\text{SNR}}(z) = \Phi_\chi(z)/\Phi_v(z) = R_{\text{SNR}}(z) R^*_{\text{SNR}}(z^{-*}) \tag{13}$$

where the spectral factor R_{SNR} is given by

$$R_{\text{SNR}}(z) = R_\chi(z) R_v^{-1}(z), \tag{14}$$

where $R_v = \sqrt{\lambda_o} H_o$ is a minimum phase spectral factor of the noise spectrum Φ_v and where R_χ is a stable spectral factor of Φ_χ, i.e.

$$R_\chi \coloneqq \begin{bmatrix} S_o R & -K S_o H_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda_o} \end{bmatrix}, \tag{15}$$

where $S_o = 1/(1 + KG_o)$ is the closed loop sensitivity function. It is straightforward to show that the predictor gradient, normalized by $\sqrt{\lambda_o}$, is given by

$$\Psi(z) = T'(z, \theta^o) R_{\text{SNR}}(z) \in \mathcal{L}_2^{n\times 2}, \tag{16}$$

where $T'(z,\theta) = \left[\frac{\partial G(z,\theta)}{\partial\theta} \;\; \frac{\partial H(z,\theta)}{\partial\theta}\right]$.

We will assume that the model parametrization is such that Ψ is stable. The stability assumption on the closed loop system and the assumptions on G_o and K imply that $R_{\text{SNR}}(z)$ and its inverse are real rational functions without poles on the unit circle and hence are $\mathcal{L}_2^{2\times 2}$ functions, as well as bounded on the unit circle.

3.2 Utilizing alternative model parametrizations

In this section we will derive an expression for the asymptotic covariance (5) of the estimate $J(\hat{\theta}_N)$ of an arbitrary differentiable quantity $J : \mathbb{R}^n \to \mathbb{C}^{1\times p}$ when Ψ in (5) is given by (16). While this can be done on a case by case basis for different model structures using Lemma 1, we will instead use an orthonormal-basis representation, (e.g., impulse response coefficients), as an intermediate parametrization in order to obtain an expression that is valid regardless of the model structure.

With the aim of representing the transfer functions in $T = [G \;\; H]$ separately, take $\{\mathcal{G}_k(z)\}_{k=1}^{\infty}$ and $\{\mathcal{H}_k(z)\}_{k=1}^{\infty}$ to be two sequences of orthonormal $\mathcal{L}_2$-functions analytic on the unit circle. For $k = 1, 2, \ldots$ define the orthonormal functions [2]

$$\mathcal{T}_{2k-1}(z) = [\mathcal{G}_k(z) \;\; 0], \;\; \mathcal{T}_{2k}(z) = [0 \;\; \mathcal{H}_k(z)]. \tag{17}$$

With $\tau = [\tau_1 \;\; \tau_2 \;\; \cdots]$, any transfer function $T = [G \;\; H]$ satisfying the assumptions in Section 3.1 can be represented by

$$T(z) = [G(z) \;\; H(z)] = \sum_{k=1}^{\infty} \tau_k \; \mathcal{T}_k(z) \tag{18}$$

on the unit circle for suitable choices of $\{\mathcal{G}_k(z)\}_{k=1}^{\infty}$ and $\{\mathcal{H}_k(z)\}_{k=1}^{\infty}$. We will assume that the sum on the right hand side of (18) has a region of convergence that includes the unit circle. For an asymptotically stable system $G(z)$ we can for example use the impulse response representation $G(z) = \sum_{k=1}^{\infty} g_k z^{-k}$ (cf. the single-sided z-transform). If $\sum_{k=1}^{\infty} |g_k| < \infty$, the sum converges uniformly to $G(z)$ on the unit circle (**?**). A more general representation is via the Takenaka-Malmquist functions

$$\mathcal{B}_k(z) \coloneqq \frac{\sqrt{1-|\xi_k|^2}}{z-\xi_k} \; \phi_{k-1}(z), \;\; k = 1, 2, \ldots \tag{19}$$

$$\phi_k(z) \coloneqq \prod_{l=1}^{k} \frac{1-\overline{\xi_l} z}{z-\xi_l}, \;\; \phi_0(z) \coloneqq 1, \tag{20}$$

which are dense in $\mathcal{H}_2$, if $\sum_{k=1}^{\infty}(1-|\xi_k|) = \infty$ for the set of pre-specified poles $\{\xi_k\}$ which are not allowed to lie on the unit circle (Ninness and Gustafsson; 1997). It is worth noticing that also unstable $G(z)$ can be represented by (18) on the unit circle, for example by a Laurent series expansion (cf. the double-sided z-transform). Since $G(z)$ is analytic on an annulus around the unit circle we can always find a series $\sum_{k=-\infty}^{\infty} a_k z^{-k}$ that converges uniformly to $G(z)$ on the unit circle. Note that also the Takenaka-Malmquist basis functions, which form a basis for $\mathcal{H}_2$, can be extended to a basis for $\mathcal{L}_2$ by appending a function $\mathcal{A}_k(z) \coloneqq \frac{1}{z}\mathcal{B}_k(1/z)$ for each basis function $\mathcal{B}_k(z)$.

By assumption, the elements of $T(z,\theta)$ are finite dimensional real rational functions with no poles on the unit circle, i.e., they can be written as $B_i(z,\theta)/A_i(z,\theta)$, $i = 1, 2$ for some polynomials B_i and A_i with real coefficients where $A_i(z,\theta)$, $i = 1, 2$, do not have any roots on the unit circle. Thus $T(z,\theta)$ belongs to $\mathcal{L}_2^2$ and hence $\tau_k(\theta)$ can be defined through the inverse transformation

$$\tau_k(\theta) := \langle T(z,\theta), \mathcal{T}_k(z) \rangle. \tag{21}$$

Then the model (11), which is parameterized by the vector θ, can also be expressed through the parametrization (18):

$$T(z,\theta) = \sum_{k=1}^{\infty} \tau_k(\theta) \mathcal{T}_k(z). \tag{22}$$

We will denote by τ the set $\{\tau_k\}_{k=1}^{\infty}$, by $\tau(\theta)$ the set $\{\tau_k(\theta)\}_{k=1}^{\infty}$, and by τ^o the set $\{\tau_k(\theta^o)\}_{k=1}^{\infty}$. Whenever we use infinite sums like in (22) we will assume that the sum converges on an annulus around the unit circle.

Lemma 2 *Under the assumptions in Section 3.1, $\tau_k(\theta)$, $k = 1, \ldots$ are differentiable at θ^o and*

$$T'(z,\theta^o) = \sum_{k=1}^{\infty} \tau_k'(\theta^o) \mathcal{T}_k(z) \in \mathcal{L}_2^{n\times 2}. \tag{23}$$

Proof By assumption $T(z,\theta)$ is continuously differentiable with respect to θ in a neighborhood of θ^o and hence the right hand side of (21) is differentiable at θ^o under the integral sign (Rudin; 1976, Theorem 9.42). Thus

$$\tau_k'(\theta^o) = \langle T'(z,\theta^o), \mathcal{T}_k(z) \rangle, \;\; k = 1, \ldots. \tag{24}$$

Now the elements of $T'(z,\theta^o)$ are given by

$$\frac{B_i'(z,\theta^o)}{A_i(z,\theta^o)} - \frac{B_i(z,\theta^o)A_i'(z,\theta^o)}{A_i^2(z,\theta^o)}, \;\; i = 1, 2 \tag{25}$$

and $T'(z,\theta^o) \in \mathcal{L}_2^{n\times 2}$ since by assumption $A_i(z,\theta^o)$, $i = 1, 2$ does not have any roots on the unit circle. Therefore,

[2] This sequence that alternates between $\mathcal{G}_k$ and $\mathcal{H}_k$ is a construction that enables us to express T with one infinite sum in (18).

(23) follows from the inverse transformation of (24). We remark that $\{\mathcal{T}_k\}$ does not have to be complete. It is only required that $T(z,\theta)$ is representable in this orthonormal system. □

Next we will link the parameter set τ to the quantity of interest J. Let $J : \mathbb{R}^n \to \mathbb{C}^{1\times p}$ be differentiable, with $J(\theta)$ defined by

$$J(\theta) = J_\tau(\tau(\theta)) \tag{26}$$

for some function J_τ.

Assumption 1 *The functions J and J_τ in (26) are such that*

a) the partial derivatives of J_τ with respect to τ_k exist at θ^o and satisfy

$$\nabla J_\tau(z) := \sum_{k=1}^{\infty} \left(\frac{\partial J_\tau(\tau^o)}{\partial \tau_k}\right)^* \mathcal{T}_k(z) \in \mathcal{L}_2^{p\times 2}, \tag{27}$$

b) the following chain rule applies:

$$J'(\theta^o) = \sum_{k=1}^{\infty} \tau_k'(\theta^o) \frac{\partial J_\tau(\tau(\theta^o))}{\partial \tau_k} \ . \tag{28}$$

Lemma 3 *Under the assumptions in Section 3.1 and Assumption 1 it holds that*

$$J'(\theta^o) = \langle \Psi, \nabla J_\tau R_{\text{SNR}}^{-*} \rangle \tag{29}$$

Proof First notice that, from (16),

$$\langle \Psi, \nabla J_\tau R_{\text{SNR}}^{-*} \rangle = \langle \Psi R_{\text{SNR}}^{-1}, \nabla J_\tau \rangle = \langle T'(z,\theta^o), \nabla J_\tau \rangle. \tag{30}$$

Due to Lemma 2, (27) and the orthonormality of $\{\mathcal{T}_k\}$ it follows that

$$\langle \Psi, \nabla J_\tau R_{\text{SNR}}^{-*} \rangle = \sum_{k=1}^{\infty} \tau_k'(\theta^o) \ \frac{\partial J_\tau(\tau(\theta^o))}{\partial \tau_k}, \tag{31}$$

which, according to assumption (28), equals $J'(\theta^o)$. □

Thus, Lemma 3 gives an explicit expression of a γ in the family (10), namely $\gamma = \nabla J_\tau R_{\text{SNR}}^{-*}$. Moreover, this particular choice of γ does not depend on the model structure used in the estimation, however, γ does in some sense depend on $\{\mathcal{T}_k\}$. In the next section we will use this choice of γ to analyze the asymptotic covariance (9).

4 Covariance analysis for SISO LTI systems

We are now ready to state the main result of this paper, which is an expression for the asymptotic covariance (9) that utilizes the model structure independent parametrization and the particular choice of γ that were introduced in the previous section.

Theorem 1 *Suppose that $J_\tau(\tau^o) \in \mathbb{C}^{1\times p}$ is estimated by $J(\hat{\theta}_N) = J_\tau(\tau(\hat{\theta}_N))$. Assume that the system and model assumptions in Section 3.1 and Assumption 1 hold. Then*

$$\begin{aligned}&\text{AsCov}\, J(\hat{\theta}_N) \\ &= \left\langle \mathbf{P}_{\mathcal{S}_\Psi}\left\{\nabla J_\tau R_{\text{SNR}}^{-*}\right\}, \mathbf{P}_{\mathcal{S}_\Psi}\left\{\nabla J_\tau R_{\text{SNR}}^{-*}\right\}\right\rangle\end{aligned} \tag{32}$$

Proof This result follows directly from Lemma 1 and Lemma 3. □

There is a certain decoupling between the property of interest J, the experimental conditions represented by $\Phi_{\text{SNR}}(e^{j\omega})$ and model structure $T(z,\theta)$ in the expression (32).

The property of interest enters the expression only through the function ∇J_τ which describes the sensitivity of the property J to changes in the transfer function T. One could interpret ∇J_τ as a Fréchet derivative (Luenberger; 1969) by viewing J as a functional of the system transfer function, i.e., $J = f(T(\cdot))$, or, if we include the θ-dependence $J(\theta) = f(T(\cdot,\theta))$. For small changes $\delta\theta$ we have $\delta J(\theta) = [\delta\theta]^{\text{T}} J'(\theta)$ and $\delta T(z) = [\delta\theta]^{\text{T}} T'(z,\theta)$. Similarly we have, by the Riesz representation theorem (Kreyszig; 1978), that $\delta f = \langle \delta T, y\rangle$ for some $y \in \mathcal{L}_2^{p\times 2}$ where y should be interpreted as the "functional derivative" $y = \dfrac{\delta f}{\delta T}$. Now consider

$$\delta J = \delta f \circ \delta T = [\delta\theta]^{\text{T}} \langle T', y\rangle = [\delta\theta]^{\text{T}} \langle \Psi, y R_{\text{SNR}}^{-*}\rangle$$

and recall that by definition

$$\delta J = [\delta\theta]^{\text{T}} J'(\theta) = [\delta\theta]^{\text{T}} \langle \Psi, \gamma\rangle = [\delta\theta]^{\text{T}} \langle \Psi, \nabla J_\tau R_{\text{SNR}}^{-*}\rangle.$$

By comparing these two expressions it can be seen that $\nabla J_\tau = y = \dfrac{\delta f}{\delta T}$. Thus, when J is viewed as a functional on T, ∇J_τ has the interpretation of a Fréchet derivative of J with respect to T .

∇J_τ is weighted by the inverse of $R_{\text{SNR}}^{-*}(z^{-*})$ where R_{SNR} is a spectral factor of $\Phi_{\text{SNR}}(z)$. Thus, Φ_{SNR}^{-1} must be small in the directions where ∇J_τ is large in order to produce an accurate estimate of J. This ratio is known from the expression (6) and can be interpreted as the frequency-wise noise to signal ratio.

The space $\mathcal{S}_\Psi$ is the span of the rows of

$$\Psi(z) = T'(z,\theta^o)\ R_{\text{SNR}}^{-1}(z) = T'(z,\theta^o) R_{\text{SNR}}(z).$$

The structure of this space is thus to a large extent determined by the model structure (through T'). However, the true system also determines $T'(z,\theta^o)$ (through θ^o) and together with the experimental conditions also acts as translation through the factor $R_{\text{SNR}}(z)$.

Furthermore, the projection only depends on the span of Ψ, i.e., the subspace $\mathcal{S}_\Psi$. From these two observations it follows that all model structures whose predictor gradients span the same space will have exactly the same asymptotic covariance.

Example 1 *Order n Laguerre models (Wahlberg; 1991) with poles in ξ will have the same asymptotic variance as fixed denominator models of order n with a pole of multiple n at ξ, since the span of Ψ is the same.*

Example 2 *The asymptotic covariance of model structures that are scaled by a constant factor remains the same, if the model structure is such that a scaling of the model structure results in a linear change in the parameters i.e., replacing $T(z,\theta)$ with $\tilde{T}(z,\tilde{\theta}) = \alpha T(z,\theta)$, results $\tilde{\theta} = \beta\theta$ (e.g. model structures that are linear in the parameters). Again, since both the function to be projected and the subspace do not change.*

On the other hand if the experimental conditions are changed so that the signal to noise ratio R_{SNR} is scaled by a factor β, then since $\mathcal{S}_\Psi$ will remain the same (even though Ψ is re-scaled), the asymptotic variance is scaled by $1/\beta^2$.

The result in Theorem 1 is basically applicable whenever the predictor gradient is given by (16) and thus very general. The expression (32) is an exact representation of the asymptotic variance (5) which is valid for a wide range of LTI model structures, including commonly used structures such as ARMAX, output-error and Box-Jenkins, and it can be used for both open loop and closed loop identification. Furthermore it expresses the variance of any property of the estimated model, provided this property can be expressed as a differentiable function of the (impulse response) coefficients τ_k satisfying the conditions in the theorem.

Theorem 1 illustrates the flexibility offered by (10). The function $\nabla J_\tau(z)\ R_{\text{SNR}}^{-*}(z^{-*})$ is a function in the set (10) of functions γ that can be used in Lemma 1 such that $\langle \Psi, \gamma \rangle$ is the sensitivity of the quantity of interest with respect to the model parameters (this is the essence of Theorem 1). However, this function is chosen with care so that it can be used regardless of the model structure (which determines $T'(z,\theta^o)$). It is due to this that the decoupling between the function of interest and the model structure, discussed above, is obtained. This also opens up the possibility to derive upper bounds that are model structure independent, which will be presented in the next section. This is one of the features offered by the geometric approach employed in this paper. The existence of such γ was previously discussed in Hjalmarsson and Mårtensson (2011), however, the explicit construction $\nabla J_\tau(z)\ R_{\text{SNR}}^{-*}(z^{-*})$ employed here is completely novel. For further discussion on the geometric approach we refer to Hjalmarsson and Mårtensson (2011).

4.1 Upper bounds

One advantage with the new expression (32) is that it is easy to provide simple, model structure independent, bounds for (9). For any closed subspace Ω such that $\mathcal{S}_\Psi \subseteq \Omega$ we can get an upper bound of (9) by projecting onto Ω instead of $\mathcal{S}_\Psi$ (Hjalmarsson and Mårtensson; 2011, Lemma II.6). The bounds presented in this paper are obtained by projection onto $\mathcal{H}_2^2$ or $\mathcal{L}_2^2$. All variables belong to $\mathcal{L}_2^2$ by assumption and then the projection onto $\mathcal{L}_2^2$ always equals the variable itself.

Theorem 2 *Let the conditions of Theorem 1 be fulfilled. An upper bound of the asymptotic covariance of $J(\hat{\theta}_N)$ is then given by*

$$\text{AsCov}\ J(\hat{\theta}_N) \leq \langle \nabla J_\tau \Phi_{\text{SNR}}^{-1}, \nabla J_\tau \rangle. \tag{33}$$

Proof The result is obtained by projection onto $\mathcal{L}_2^2$ instead of $\mathcal{S}_\Psi$ in (32), cf. Hjalmarsson and Mårtensson (2011, Lemma II.6). □

The bounds in Theorem 2 typically (but not always) depend on the true underlying system through ∇J_τ and Φ_{SNR}. However, notice that they do not depend on the model structure. This means that they are valid for *any* model structure (under the model assumptions in Section 3.1), which also means that they apply to arbitrarily high model orders.

When we are projecting onto $\mathcal{L}_2^2$, i.e., when the projection is removed, the bounds derived are typically conservative even as the model order increases since $\mathcal{S}_\Psi \subseteq \mathcal{H}_2^2$ regardless of the model order and model structure, while the function that is projected, $\nabla J_\tau\, R_{\text{SNR}}^{-*}$, typically has a term that belongs to the complement of $\mathcal{H}_2^2$. This conservativeness will be further investigated in the next section.

4.2 Conservativeness

The upper bounds derived in Section 4.1 are model structure independent and in general conservative. In this section we will try to investigate how conservative they are. We will proceed by deriving also a lower bound on $\text{AsCov}\ J(\hat{\theta}_N)$ in order to bound the conservativeness. To this end, define

$$\begin{aligned} \eta_u &\coloneqq \sup_\omega \lambda_{\max}\left\{\Phi_{\text{SNR}}\right\} \\ \eta_l &\coloneqq \inf_\omega \lambda_{\min}\left\{\Phi_{\text{SNR}}\right\}, \end{aligned} \tag{34}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalue of Φ_{SNR}, respectively.

Lemma 4 *Let the conditions of Theorem 1 be fulfilled. Bounds of the asymptotic covariance of $J(\hat{\theta}_N)$ are then given by*

$$\begin{aligned}&\frac{1}{\eta_u}\langle \mathbf{P}_{\mathcal{S}_{T'}}\{\nabla J_\tau\}, \mathbf{P}_{\mathcal{S}_{T'}}\{\nabla J_\tau\}\rangle \\ &\leq \text{AsCov}\, J(\hat{\theta}_N) \\ &\leq \frac{1}{\eta_l}\langle \mathbf{P}_{\mathcal{S}_{T'}}\{\nabla J_\tau\}, \mathbf{P}_{\mathcal{S}_{T'}}\{\nabla J_\tau\}\rangle \end{aligned} \quad (35)$$

Proof We expand the projection in (32) and get

$$\begin{aligned}&\text{AsCov}\, J(\hat{\theta}_N) \\ &= \langle \nabla J_\tau R_{\text{SNR}}^{-*}, \Psi\rangle \langle \Psi, \Psi\rangle^{-1} \langle \Psi, \nabla J_\tau R_{\text{SNR}}^{-*}\rangle \\ &= \langle \nabla J_\tau, T'\rangle \langle \Psi, \Psi\rangle^{-1} \langle T', \nabla J_\tau\rangle. \end{aligned} \quad (36)$$

Then, since $\langle \Psi, \Psi\rangle = \langle T'\Phi_{\text{SNR}}, T'\rangle$, we have that

$$\eta_l \langle T', T'\rangle \leq \langle \Psi, \Psi\rangle \leq \eta_u \langle T', T'\rangle. \quad (37)$$

Applying these bounds to (36) the Theorem follows. □

As we will see in Section 6, for some particular choices of J, we have that $\nabla J_\tau \in \mathcal{S}_{T'}$ and in those cases we can remove the projection. Furthermore, we can expect that $\mathcal{S}_{T'}$ spans $\mathcal{H}_2^2$ when the model order increases, and since $\nabla J_\tau \in \mathcal{H}_2^2$, we can expect that $\nabla J_\tau \in \mathcal{S}_{T'}$ asymptotically in the model order. In the case when $\nabla J_\tau \in \mathcal{S}_{T'}$, we have the following theorem.

Theorem 3 *Let the conditions of Theorem 1 be fulfilled and let $\nabla J_\tau \in \mathcal{S}_{T'}$. A lower bound of the asymptotic covariance of $J(\hat{\theta}_N)$ is then given by*

$$\frac{\eta_l}{\eta_u}\langle \nabla J_\tau \Phi_{\text{SNR}}^{-1}, \nabla J_\tau\rangle \leq \text{AsCov}\, J(\hat{\theta}_N), \quad (38)$$

where η_l and η_u are defined in (34).

Proof Similarly to (37), we also have that

$$\langle \nabla J_\tau \Phi_{\text{SNR}}^{-1}, \nabla J_\tau\rangle \leq \frac{1}{\eta_l}\langle \nabla J_\tau, \nabla J_\tau\rangle. \quad (39)$$

Thus, in the case when $\nabla J_\tau \in \mathcal{S}_{T'}$, combining (39) and (35) gives the lower bound. □

Comparing Theorem 3 with Theorem 2, we see that the only difference between the lower bound and upper bound is the scaling factor η_l/η_u. When the spectrum of the signal-to-noise ratio is rather flat we expect this ratio to be closer to one, and thus expect the bound in Theorem 2 to be less conservative.

4.3 Curse of Complexity

In this section we will compare the results based on Theorem 1 with the asymptotic covariance expressions implied by the results in Ljung (1985) that are asymptotic in model order.

Using the orthonormality of $\{\mathcal{T}_k\}$ in the representation (18) gives that $\tau_l = \langle T, \mathcal{T}_l\rangle$ and hence (with $\tilde{\tau}_k \coloneqq \tau_k(\hat{\theta}_N) - \tau_k^o$, $\widetilde{T}(z) \coloneqq T(z, \hat{\theta}_N) - T_o(z)$ and with m being the model order)

$$\begin{aligned}&\lim_{m\to\infty} \frac{1}{m}\, \mathbf{E}\,[\tilde{\tau}_k^* \tilde{\tau}_l] \\ &= \lim_{m\to\infty} \frac{1}{m}\, \mathbf{E}\left[\left\langle \mathcal{T}_k(z), \widetilde{T}(z)\right\rangle \left\langle \widetilde{T}(\zeta), \mathcal{T}_l(\zeta)\right\rangle\right] \\ &= \lim_{m\to\infty} \frac{1}{m}\, \mathbf{E}\left[\left\langle \left\langle \mathcal{T}_k(z), \widetilde{T}^*(\zeta)\widetilde{T}(z)\right\rangle, \mathcal{T}_l(\zeta)\right\rangle\right] \\ &= \left\langle \left\langle \mathcal{T}_k(z), \lim_{m\to\infty} \frac{1}{m}\mathbf{E}\left[\widetilde{T}^*(\zeta)\widetilde{T}(z)\right]\right\rangle, \mathcal{T}_l(\zeta)\right\rangle, \end{aligned} \quad (40)$$

assuming that the limit operation and the integration commute. If we now use the asymptotic result

$$\lim_{m\to\infty} \frac{1}{m}\text{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N) = \Phi_{\text{SNR}}^{-1}(\mathrm{e}^{\mathrm{j}\omega}) \quad (41)$$

derived in Ljung (1985) and another result from Ljung (1985), namely that frequency function estimates at different frequencies become uncorrelated as the model order $m \to \infty$, (40) collapses to zero, which in turn suggests that for any J of the type described in Theorem 1

$$\lim_{m\to\infty} \frac{1}{m}\, \text{AsCov}\, J(\hat{\theta}_N) = 0. \quad (42)$$

We have thus obtained, by direct application of the model order asymptotic results in (Ljung; 1985) that the asymptotic variance is of the order $o(m)$. Note that $\text{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N)$ grows unbounded with m as in (41), which seems to contradict (42), but $T(\mathrm{e}^{\mathrm{j}\omega})$ does not belong to the allowed class of functions J in Theorem 1. This situation will be addressed later in Section 6.1.

The result (42) is considerably weaker than the upper bound (33) derived in this paper. Thus we have shown that, when the conditions of Theorem 1 hold, the upper bounds derived in this paper are significantly more accurate expressions for the asymptotic covariance than the asymptotic covariance expressions implied by the results in Ljung (1985).

In order to go from the results in Ljung (1985) and arrive at something similar to (33) in Theorem 2 we must remove the scaling factor $1/m$ and interpret (41) as

$$\lim_{m\to\infty} \mathbf{E}\left[\widetilde{T}^*(\xi)\widetilde{T}(z)\right] = \Phi_{\text{SNR}}^{-1}(z)\delta(z-\xi), \quad (43)$$

with $\delta(x)$ being the Dirac delta. Note that Ljung (1985) does *not* give support for this interpretation. Removing the factor $1/m$ in (40) and then using (43) yields

$$\begin{aligned}\lim_{m\to\infty} \mathbf{E}\left[\tilde{\tau}_k^* \tilde{\tau}_l\right] &= \left\langle \left\langle \mathcal{T}_k(z), \Phi_{\text{SNR}}^{-1}(z)\delta(z-\xi)\right\rangle, \mathcal{T}_l(\zeta)\right\rangle \\ &= \left\langle \mathcal{T}_k, \mathcal{T}_l \Phi_{\text{SNR}}^{-1}\right\rangle\end{aligned}$$

and with J, J_τ and ∇J_τ as in Theorem 1 we get

$$\lim_{m\to\infty} \text{AsCov}\, J(\hat{\theta}_N) = \left\langle \nabla J_\tau \Phi_{\text{SNR}}^{-1}, \nabla J_\tau \right\rangle,$$

which is the same as the upper bound (33).

The fact that the scaling factor $1/m$ is not present is especially important as it shows that certain properties, even of highly complex systems, are not subject to what is known as the "curse of complexity", i.e., there are system properties that can be accurately identified using full order models also when the system is highly complex. In Section 6 we will see some examples of such properties. For more details on this important topic we refer the reader to Hjalmarsson (2005); Rojas et al. (2008, 2010); Mårtensson and Hjalmarsson (2009).

5 Explicit use of orthonormal basis functions

In this section we will show how to explicitly express the asymptotic variance (5) in terms of an orthonormal basis for $\mathcal{S}_\Psi$. For any given $\Psi \in \mathcal{L}_2^{n\times 2}$ an orthonormal basis $\{\mathcal{B}_k\}_{k=1}^r$, $r \le n$ for $\mathcal{S}_\Psi$ can be constructed by, e.g., Gram-Schmidt orthonormalization. In some cases it is possible to derive explicit expressions for the basis functions $\mathcal{B}_k$. A well known case (Ninness and Gustafsson; 1997) is when

$$\mathcal{S}_\Psi = \text{Span}\left\{\frac{z^{-1}}{L(z)}, \frac{z^{-2}}{L(z)}, \dots, \frac{z^{-n}}{L(z)}\right\}, \tag{44}$$

where $L(z) = \prod_{k=1}^{n_l}(1-\xi_k z^{-1})$, $|\xi_k| < 1$ for some set of specified poles $\{\xi_1, \dots, \xi_{n_l}\}$ and where $n \ge n_l$. Then, it holds that

$$\mathcal{S}_\Psi = \text{Span}\left\{\mathcal{B}_1, \dots, \mathcal{B}_n\right\},$$

where $\{\mathcal{B}_k\}$ are the Takenaka-Malmquist functions given by (19)–(20), with $\xi_k = 0$ for $k = n_l+1, \dots, n$. In Ninness and Hjalmarsson (2004) it is shown that the structure (44) holds for common model structures such as Output-Error and Box-Jenkins provided the input spectrum has no zeros and sufficiently many numerator coefficients are estimated. Notice that the system zeros do not affect the basis functions above.

5.1 Explicit variance expressions

As usual, let $\Psi \in \mathcal{L}_2^{n\times 2}$ and $\Lambda \in \mathbb{C}^{n\times p}$. If $\{\mathcal{B}_k\}_{k=1}^r$ is an orthonormal basis for $\mathcal{S}_\Psi$ it is straightforward to show that

$$\Psi^*(z)\langle \Psi, \Psi\rangle^\dagger \Psi(z) = \sum_{k=1}^r \mathcal{B}_k^*(z)\mathcal{B}_k(z).$$

Then, if $\Lambda = \Psi(z_o)L$ for some $z_o \in \mathbb{C}$ and $L \in \mathbb{C}^{2\times p}$ we get that (5) can be written as

$$\text{AsCov}\, J(\hat{\theta}_N) = L^* \sum_{k=1}^r \mathcal{B}_k^*(z_o)\mathcal{B}_k(z_o) L, \tag{45}$$

which also is found in Hjalmarsson and Mårtensson (2011). The next result is an adaptation of (45) to the re-parametrization in Theorem 1.

Theorem 4 *Let the assumptions in Theorem 1 hold and let $\{\mathcal{B}_k\}_{k=1}^r$, $r \le n$, be an orthonormal basis for $\mathcal{S}_\Psi$. Assume also that*

$$\frac{\partial J_\tau(\tau(\theta^o))}{\partial \tau_k} = \mathcal{T}_k(z_o)\alpha \tag{46}$$

for some $\alpha \in \mathbb{C}^{2\times p}$ and $z_o \in \mathbb{C}$. Then (45) holds with

$$L = R_{\text{SNR}}^{-1}(z_o)\alpha. \tag{47}$$

Proof We need to prove that

$$\Lambda = \langle \Psi, \nabla J_\tau R_{\text{SNR}}^{-*}\rangle = \Psi(z_o) R_{\text{SNR}}^{-1}(z_o)\alpha. \tag{48}$$

From (46) it holds that

$$\begin{aligned}\langle \Psi, \nabla J_\tau R_{\text{SNR}}^{-*}\rangle &= \sum_{l=1}^\infty \langle \Psi, \mathcal{T}_l R_{\text{SNR}}^{-*}\rangle\ \mathcal{T}_l(z_o)\ \alpha \\ &= \sum_{l=1}^\infty \langle \Psi R_{\text{SNR}}^{-1}, \mathcal{T}_l\rangle\ \mathcal{T}_l(z_o)\ \alpha \\ &= \mathbf{P}_{\mathcal{Y}}\{\Psi R_{\text{SNR}}^{-1}\}(z_o)\ \alpha,\end{aligned}$$

where $\mathcal{Y}$ is the space spanned by $\{\mathcal{T}_l\}_{l=1}^\infty$. However, due to (16) and (23), $\Psi R_{\text{SNR}}^{-1} \in \mathcal{Y}$, so the projection can be removed giving (48). □

Notice that under the conditions in Theorem 4, the condition that $\nabla J_\tau R_{\text{SNR}}^{-*}(z^{-*}) \in \mathcal{L}_2^{p\times 2}$ in (27) can be written as

$$\alpha^* \sum_{k=1}^\infty \mathcal{T}_k^*(z_o)\mathcal{T}_k(z)\ R_{\text{SNR}}^{-*}(z^{-*}) \in \mathcal{L}_2^{p\times 2}.$$

5.2 Upper bounds

Here we will describe a case when a simple bound for (45) can be found by a projection onto the subspace $\mathcal{H}_2^2 \subset \mathcal{L}_2^2$. This gives a tighter bound than Theorem 2, where the projection was made onto $\mathcal{L}_2^2$.

Let the asymptotic variance be given by (45) for a z_o such that $|z_o| > 1$. An upper bound of (45) is obtained by exchanging the basis functions $\{\mathcal{B}_k\}$ for basis functions $\{\widetilde{\mathcal{B}}_k\}$ for the larger subspace $\mathcal{H}_2^2$. Since all elements of both $G'(z, \theta^o)$ and $H'(z, \theta^o)$ have at least one time delay, all elements of Ψ will also have at least one time delay, and therefore we will exclude constant functions. One such orthonormal basis is given by $\{\widetilde{\mathcal{B}}_k(z)\}_{k=1}^{\infty}$ where $\widetilde{\mathcal{B}}_k(z) = [z^{-(k+1)/2}\ 0]$ when k is odd, and $\widetilde{\mathcal{B}}_k(z) = [0\ z^{-k/2}]$ when k is even. For $|z_o| > 1$ we then get

$$\sum_{k=1}^{\infty} \overline{\widetilde{\mathcal{B}}_k}^*(z_o)\widetilde{\mathcal{B}}_k(z_o) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sum_{k=1}^{\infty} |z_o|^{-2k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{|z_o|^2 - 1},$$

which when inserted in (45) gives the upper bound

$$\text{AsCov}\, J(\hat{\theta}_N) \leq \frac{1}{|z_o|^2 - 1}\ \alpha^*\ \Phi_{\text{SNR}}^{-1}(z_o)\ \alpha. \tag{49}$$

In some cases, we can find a similar expression that holds with equality, as we will see in the following section.

5.3 Model structure independent example

Now, we turn to a less obvious insight that does not seem to be generally known. Suppose that the explicit expression (45) holds for some z_o strictly outside the unit circle and that Ψ contains a pole at z_o^{-1}. Suppose further that the orthonormal basis used in (45) is of the form (19). If we then order the poles in Ψ such that $\xi_1 = z_o^{-1}$, we obtain from (19) that $\mathcal{B}_1(z_o) = \sqrt{1 - |z_o|^{-2}}/(z_o - z_o^{-1})$ and $\mathcal{B}_k(z_o) = 0$, $k = 2, \ldots, n$, resulting in that

$$\text{AsCov}\, J(\hat{\theta}_N) = \frac{1 - |z_o|^{-2}}{|z_o - z_o^{-1}|^2}\ \alpha^*\ \Phi_{\text{SNR}}^{-1}(z_o)\ \alpha. \tag{50}$$

This expression is remarkable in that it is independent of the model structure and model order. Now recall that $\Psi(z) = T'(z, \theta^o)R_{\text{SNR}}(z_o)$. Thus in the cases when (45) holds and when the experimental conditions can be chosen such that $R_{\text{SNR}}(z)$ has a pole at z_o^{-1}, this choice makes the asymptotic covariance the same for different model structures and arbitrary model order. This insight is important in order to come to terms with the so called "curse of dimensionality" discussed in Section 4.3. We will illustrate this idea in Section 6.4 where the objective is to identify NMP-zeros. The geometric approach has been used in Hjalmarsson et al. (2006); Mårtensson and Hjalmarsson (2009) to generalize this result as well as to show that certain optimality properties also hold from an experiment design perspective.

6 Analysis of some LTI system properties

In this section we apply the results from Sections 4 and 5 to some specific examples of the function $J(\theta)$. The main purpose is to show how the geometric approach can be used in the analysis, but each of these results are also of independent interest. Some of the examples does not depend on the noise model. For those results we introduce the notation

$$\nabla J_\tau(z) =: \left[\nabla J_\tau^g(z)\ \ \nabla J_\tau^h(z)\right]. \tag{51}$$

6.1 Frequency response

We will first look at the covariance of the frequency response estimate, i.e., $J(\theta) = T(e^{j\omega_o}, \theta)$ for a fixed frequency ω_o when T_o is stable (so that the frequency response is well defined). Then we get

$$\Lambda = T'(e^{j\omega_o}, \theta^o) = \Psi(e^{j\omega_o})R_{\text{SNR}}^{-1}(e^{j\omega_o}),$$

where Ψ is given by (16). Now (45) can be applied to get the covariance expression

$$\begin{aligned}&\text{AsCov}\, T(e^{j\omega_o}, \hat{\theta}_N) \\ &\quad = R_{\text{SNR}}^{-*}(e^{j\omega_o}) \sum_{k=1}^{n} \mathcal{B}_k^*(e^{j\omega_o})\mathcal{B}_k(e^{j\omega_o})\ R_{\text{SNR}}^{-1}(e^{j\omega_o}),\end{aligned} \tag{52}$$

where $\{\mathcal{B}_k\}_{k=1}^n$ is any orthonormal basis for the space $\mathcal{S}_\Psi$.

It is instructive to also consider the formulation in Theorem 1 for expressing $\text{AsCov}\, T(e^{j\omega_o}, \hat{\theta}_N)$. We will use basis functions $\mathcal{T}_k$ (17) defined by $\mathcal{G}_k(z) = \mathcal{H}_k(z) = z^{-(k-1)}$ so that

$$T(z, \theta) = \sum_{k=1}^{\infty} \tau_k(\theta)\mathcal{T}_k(z)$$

is parameterized in terms of the impulse responses of $G(z, \theta)$ and $H(z, \theta)$. For this problem $J_\tau(\tau(\theta)) = \sum_{k=1}^{\infty} \tau_k(\theta)\mathcal{T}_k(e^{j\omega_o})$ and hence $\partial J_\tau(\tau)/\partial \tau_k = \mathcal{T}_k(e^{j\omega_o})$ which implies that ∇J_τ, defined in (27), is not an $\mathcal{L}_2$-function. Thus we instead look at $J(\theta) = T(z_o, \theta)$, $z_o = re^{j\omega_o}$, $r > 1$ and later we let $r \to 1$. The function ∇J_τ is now given by

$$\nabla J_\tau(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sum_{k=1}^{\infty} \bar{z}_o^{-k} z^{-k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\bar{z}_o^{-1} z^{-1}}{1 - \bar{z}_o^{-1} z^{-1}},$$

which is a function in $\mathcal{L}_2$ so that (27) holds. Furthermore, (28) follows from Lemma 2 with $z = \mathrm{e}^{\mathrm{j}\omega_o}$ for this problem and hence Assumption 1 holds. Thus, Theorem 1 applies under the assumptions in Section 3.1.

In order to calculate the projection $\mathbf{P}_{\mathcal{S}_\Psi}\{\nabla J_\tau R_{\text{SNR}}^{-*}\}$ in Theorem 1, let $\{\mathcal{B}_k\}_{k=1}^n$ be an orthonormal basis for $\mathcal{S}_\Psi$, then

$$\begin{aligned}&\langle \nabla J_\tau(z) R_{\text{SNR}}^{-*}(z^{-*}), \mathcal{B}_k(z)\rangle \\ &\quad = \frac{1}{2\pi j}\oint_{|z|=1} \frac{\bar{z}_o^{-1} z^{-1}}{1-\bar{z}_o^{-1}z^{-1}} R_{\text{SNR}}^{-*}(z^{-*})\mathcal{B}_k^*(z^{-*})\frac{\mathrm{d}z}{z} \\ &\quad = R_{\text{SNR}}^{-*}(z_o)\mathcal{B}_k^*(z_o). \end{aligned} \tag{53}$$

In the second equality we use that $z^{-1}R_{\text{SNR}}^{-*}(z^{-*})\mathcal{B}_k^*(z^{-*})$ has all poles outside the unit circle (since all $\mathcal{B}_k$ contain at least one unit time delay which cancels the factor z^{-1}) and residue calculus, see e.g., Wunsch (1993), gives the result (53). The projection $\mathbf{P}_{\mathcal{S}_\Psi}\{\nabla J_\tau R_{\text{SNR}}^{-*}\}$ can now be computed as

$$\begin{aligned}&\mathbf{P}_{\mathcal{S}_\Psi}\{\nabla J_\tau R_{\text{SNR}}^{-*}\} \\ &\quad = \sum_{k=1}^n \langle \nabla J_\tau R_{\text{SNR}}^{-*}, \mathcal{B}_k\rangle \mathcal{B}_k = R_{\text{SNR}}^{-*}(z_o)\sum_{k=1}^n \mathcal{B}_k^*(z_o)\mathcal{B}_k.\end{aligned}$$

With $z_o = r\mathrm{e}^{\mathrm{j}\omega_o}$ and $r \to 1$, we get the asymptotic covariance

$$\begin{aligned}&\text{AsCov}\, T(\mathrm{e}^{\mathrm{j}\omega_o}, \hat{\theta}_N) \\ &= R_{\text{SNR}}^{-*}(\mathrm{e}^{\mathrm{j}\omega_o})\sum_{k=1}^n\sum_{p=1}^n \mathcal{B}_k^*(\mathrm{e}^{\mathrm{j}\omega_o})\langle \mathcal{B}_k, \mathcal{B}_p\rangle \mathcal{B}_p(\mathrm{e}^{\mathrm{j}\omega_o})\; R_{\text{SNR}}^{-1}(\mathrm{e}^{\mathrm{j}\omega_o}) \\ &= R_{\text{SNR}}^{-*}(\mathrm{e}^{\mathrm{j}\omega_o})\sum_{k=1}^n \mathcal{B}_k^*(\mathrm{e}^{\mathrm{j}\omega_o})\mathcal{B}_k(\mathrm{e}^{\mathrm{j}\omega_o})\; R_{\text{SNR}}^{-1}(\mathrm{e}^{\mathrm{j}\omega_o}),\end{aligned}$$

which, of course, is the same as (52).

The covariance expression (52) was first established in Ninness and Hjalmarsson (2004) by employing the theory of reproducing kernels, see also Ninness and Hjalmarsson (2005). Above we have shown that the results in Ninness and Hjalmarsson (2004) can be given an alternative system theoretic interpretation as resulting from a projection of the weighted z-transform of the sensitivity of the system frequency function with respect to the impulse response on a subspace determined by the model structure, the true system and the experimental conditions. The weighting function depends on the noise to signal ratio during the experiment (which in turn depends on the experimental conditions and the true system). Our paper can also be seen as an extension of the work in Ninness and Hjalmarsson (2004) regarding variance analysis in frequency function estimation to general quantities J.

6.2 Impulse response

In this example we look at the asymptotic variance of the coefficients τ_k of the estimated model $T(z,\theta) = \sum_{k=1}^\infty \tau_k(\theta)\mathcal{T}_k(z)$, cf. (22), but we assume that only the first n_τ coefficients are of interest and we let

$$J_\tau(\tau) = \tau = \begin{bmatrix}\tau_1 & \cdots & \tau_{n_\tau}\end{bmatrix}.$$

Now $\frac{\mathrm{d}J_\tau}{\mathrm{d}\tau} = I$ (the identity matrix) and hence

$$\nabla J_\tau(z) = \begin{bmatrix}\mathcal{T}_1(z) \\ \vdots \\ \mathcal{T}_{n_\tau}(z)\end{bmatrix}.$$

It is straightforward to verify the chain rule, (28) of Assumption 1. Using Theorem 1, the asymptotic covariance can be expressed as

$$\begin{aligned}\text{AsCov}\,\tau(\hat{\theta}_N) &= \langle \mathbf{P}_{\mathcal{S}_\Psi}\{\nabla J_\tau R_{\text{SNR}}^{-*}\}, \mathbf{P}_{\mathcal{S}_\Psi}\{\nabla J_\tau R_{\text{SNR}}^{-*}\}\rangle \\ &\le \langle \nabla J_\tau \Phi_{\text{SNR}}^{-1}, \nabla J_\tau\rangle,\end{aligned} \tag{54}$$

where the inequality comes from Theorem 2. If we consider the impulse response coefficients g_k corresponding to $\mathcal{G}_k(z) = z^{-k}$ in (17) we get for the diagonal block of (54) that

$$\text{AsCov} g_k(\hat{\theta}_N) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_v(\omega)}{\Phi_u^r(\omega)}\,\mathrm{d}\omega.$$

6.3 $\mathcal{L}_2$-norm

Now we consider the asymptotic variance of the $\mathcal{L}_2$-norm of the estimated model $G(z,\theta) = \sum_{k=1}^\infty g_k(\theta)\mathcal{G}_k(z)$, cf. (18). The $\mathcal{L}_2$-norm is given by

$$\|G(\cdot,\theta)\| = \sqrt{\langle G(\cdot,\theta), G(\cdot,\theta)\rangle} = \sqrt{\sum_{k=1}^\infty g_k^2(\theta)}$$

and since J is independent of the noise model H, $\nabla J_\tau^h(z) = 0$ in (51) the function $\nabla J_\tau^g(z)$ is

$$\nabla J_\tau^g(z) = \sum_{k=1}^\infty \frac{g_k(\theta^o)\mathcal{G}_k(z)}{\sqrt{\sum_{l=1}^\infty g_l^2(\theta^o)}} = \frac{G_o(z)}{\|G_o\|} \in \mathcal{L}_2.$$

It is straightforward to verify the chain rule (28) in Assumption 1. Thus we can use Theorem 1 to express the asymptotic variance as

$$\text{AsCov}\,\|G(\cdot,\hat{\theta}_N)\| = \frac{\left\|\mathbf{P}_{\mathcal{S}_\Psi}\left\{\begin{bmatrix}\frac{\sqrt{\lambda_o}G_o H_o^*}{S_o^* R^*} & 0\end{bmatrix}\right\}\right\|^2}{\|G_o\|^2}. \tag{55}$$

The projection in (55) may be cumbersome to calculate, but we can use Theorem 1 to get an upper bound of the asymptotic variance:

$$\text{AsCov}\,\|G(\cdot,\hat{\theta}_N)\| \leq \frac{\|G_o\|^2_{\Phi_v/\Phi_u^r}}{\|G_o\|^2}.$$

The bound can also be written in the form

$$\text{AsCov}\,\|G(\cdot,\hat{\theta}_N)\| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Phi_v(\omega)}{\Phi_u^r(\omega)}w_G(\omega)\,\mathrm{d}\omega,$$

where $w_G(\omega) = |G_o(\mathrm{e}^{\mathrm{j}\omega})|^2/\|G_o\|^2$ is a weighting function with $\frac{1}{2\pi}\int_{-\pi}^{\pi} w_G(\omega)\,\mathrm{d}\omega = 1$. The weighting function $w_G(\omega)$ gives more weight to frequencies where the gain is large. It is also straightforward to provide a lower bound by following the derivations in Section 4.2. Provided $G_o(z) \in \mathcal{S}_T'$, which holds for many model structures, a lower bound can be given as

$$\frac{\inf_\omega \lambda_{\min}\left\{\Phi_{\text{SNR}}^{-1}\right\}}{\sup_\omega \lambda_{\max}\left\{\Phi_{\text{SNR}}^{-1}\right\}}\frac{\|G_o\|^2_{\Phi_v/\Phi_u^r}}{\|G_o\|^2} \leq \text{AsCov}\,\|G(\cdot,\hat{\theta}_N)\|.$$

A simple upper bound is given by

$$\text{AsCov}\,\|G(\cdot,\hat{\theta}_N)\| \leq \sup_\omega \frac{\Phi_v(e^{j\omega})}{\Phi_u^r(e^{j\omega})},$$

which could perhaps be of value for practitioners, since it is independent of the true system. For example, if Φ_v is known, without prior knowledge of G_o, Φ_u^r can be designed to guarantee a bound on the covariance of the $\mathcal{L}_2$-norm of the estimated model $G(z,\theta)$.

6.4 Non-minimum phase zeros

Next we consider estimation of NMP-zeros of a stable system G_o. The zeros of the system are defined as the solutions z to the equation $G(z,\theta) = \sum_{k=1}^{\infty} g_k(\theta)z^{-k} = 0$ and we assume that the zero of interest, z_o, is non-minimum phase, i.e., $|z_o| > 1$. The quantity of interest is thus $J(\hat{\theta}_N) = z_o(\hat{\theta}_N)$. Since J is independent of the noise model H, $\nabla J_\tau^h(z) = 0$ in (51). Similar to Mårtensson and Hjalmarsson (2009) we obtain

$$\frac{\partial J_\tau^g(\tau^o)}{\partial g_k} = -\frac{z_o}{\widetilde{G}_o(z_o)}z_o^{-k},$$

where $\widetilde{G}_o(z) = G_o(z)/(1-z_o z^{-1})$, which gives, for $|z| > 1/|z_o|$, that

$$\begin{aligned}\nabla J_\tau^g(z) &= -\frac{\bar{z}_o}{\overline{\widetilde{G}_o(z_o)}}\sum_{k=1}^{\infty}\bar{z}_o^{-k}z^{-k}\\ &= -\frac{\bar{z}_o}{\overline{\widetilde{G}_o(z_o)}}\frac{\bar{z}_o^{-1}z^{-1}}{(1-\bar{z}_o^{-1}z^{-1})},\end{aligned}$$

which is in $\mathcal{L}_2$. It is straightforward to verify that the chain rule (28) of Assumption 1 hold and Theorem 1 thus applies. Suppose, for simplicity, that the numerator polynomial $B(\mathrm{q},\theta)$, in $G(\mathrm{q},\theta) = B(\mathrm{q},\theta)/A(\mathrm{q},\theta)$, is independently parameterized of $A(\mathrm{q},\theta)$. Lemma 2 gives that $g_k'(\theta^o) = \langle G'(z,\theta^o), z^{-k}\rangle$ so that

$$\begin{aligned}\sum_{k=1}^{\infty} g_k'(\theta^o)\,\frac{\partial J_\tau^g(\tau^o)}{\partial g_k} &= -\frac{z_o}{\widetilde{G}_o(z_o)}\,\sum_{k=1}^{\infty}\langle G'(z,\theta^o), z^{-k}\rangle\, z_o^{-k}\\ &= -\frac{z_o}{\widetilde{G}_o(z_o)}\,G'(z_o,\theta^o) = -\frac{z_o}{\widetilde{B}_o(z_o)}\,\begin{bmatrix}B'(z_o(\theta^o),\theta^o)\\ 0\end{bmatrix},\end{aligned}$$

where $\widetilde{B}_o(z) = B(z,\theta^o)/(1-z_o z^{-1})$. The last expression then equals $J'(\theta^o)$ (Mårtensson and Hjalmarsson; 2009).

We have thus verified the conditions in Theorem 1 and the asymptotic variance can be calculated in the same way as in (53)-(54), or alternatively (45) could be applied, and, without giving all details, we get

$$\text{AsCov}\,z_o(\hat{\theta}_N) = \frac{\lambda_0|z_o|^2}{|\widetilde{G}_o(z_o)|^2}\frac{|H_o(z_o)|^2}{|R(z_o)|^2}\sum_{k=1}^{n}|\mathcal{B}_k^1(z_o)|^2, \quad (56)$$

where $\mathcal{B}_k := [\mathcal{B}_k^1, \mathcal{B}_k^2]$ and $\{\mathcal{B}_k\}_{k=1}^n$ is an orthonormal basis for $\mathcal{S}_\Psi$. The expression (56) is used in Mårtensson and Hjalmarsson (2009) where also explicit expressions and bounds for $\sum_{k=1}^n|\mathcal{B}_k^1(z_o)|^2$ are derived for certain model structures. When an orthonormal basis of the type given in Section 5 can be used, we see from (56) that the asymptotic variance for a NMP-zero will be large if there is another NMP-zero nearby. This follows from that the factor $|\widetilde{G}_o(z_o)|^2$ in the denominator will be small in this case and that the basis functions are independent of the system zeros (see Section 5).

Bounds on the asymptotic variance for NMP zeros can be derived using (49):

$$\begin{aligned}&\text{AsCov}\,z_o(\hat{\theta}_N)\\ &\quad\leq \frac{\lambda_0}{(1-|z_o|^{-2})|\widetilde{G}_o(z_o)|^2}\frac{|H_o(z_o)|^2}{|S_o(z_o)R(z_o)|^2}, \qquad (57)\end{aligned}$$

or using Theorem 2:

$$\begin{aligned}\text{AsCov}\,z_o(\hat{\theta}_N) &\leq \frac{1}{|\widetilde{G}_o(z_o)|^2}\left\|\frac{1}{(1-\bar{z}_o^{-1}z^{-1})}\right\|^2_{\Phi_v/\Phi_u^r}\\ &= \frac{\lambda_0}{(1-|z_o|^{-2})|\widetilde{G}_o(z_o)|^2}\left\|\frac{H_o}{S_oR}\right\|^2_{w_Z}, \quad (58)\end{aligned}$$

where $w_Z(\omega) = (1-|z_o|^{-2})/|1-z_o^{-1}\mathrm{e}^{\mathrm{j}\omega}|^2$ can be seen as a weighting function with $\frac{1}{2\pi}\int_{-\pi}^{\pi} w_Z(\omega)\mathrm{d}\omega = 1$. The weighting function $w_Z(\omega)$ focuses on frequencies around $\omega = \arg(z_o)$. Good accuracy of an NMP-zero estimate is

thus guaranteed if $|H_o/(S_oR)|$ is small in this frequency range.

The bound (58) is always larger than the bound (57) which is a tight bound in the sense that equality holds (in many cases) when the model order n goes to infinity, see Mårtensson and Hjalmarsson (2009), which provides a quite complete asymptotic variance analysis of both zero and pole estimates. It is also shown in Mårtensson and Hjalmarsson (2009) that the convergence is exponentially fast in the model order, in some cases.

Before closing this section, we illustrate the idea outlined at the end of Section 3, i.e., that the input can be used to make the asymptotic variance the same for different model structures and arbitrary model orders. For simplicity we will assume that the NMP zero is real and that the system is operating in open loop so that the prediction error gradient (16) is given by

$$\Psi(z) = T'(z,\theta^o) \begin{bmatrix} \frac{R(z)}{\sqrt{\lambda_o}\, H_o(z)} & 0 \\ 0 & \frac{1}{H_o(z)} \end{bmatrix}.$$

Assume first that an output error model is used and that the spectral factor of the input is chosen as $R(z) = 1/(z - z_o^{-1})$. When $n_b \geq n_f + 1$, as discussed in Section 5, it then follows that (19) is an orthonormal basis with $\xi_1 = z_o^{-1}$ and the other poles being the poles of the true system dynamics are counted twice. Similar to (50), (56) then collapses to

$$\begin{aligned} &\text{AsCov}\, z_o(\hat{\theta}_N) \\ &= \frac{\lambda_0 |z_o|^2}{|\widetilde{G}_o(z_o)|^2} \frac{|H_o(z_o)|^2}{|R(z_o)|^2} \sum_{k=1}^{n} |\mathcal{B}_k^1(z_o)|^2 = \frac{\lambda_0\, (|z_o|^2 - 1)}{|\widetilde{G}_o(z_o)|^2}. \end{aligned}$$

Due to the presence of $R(z)$ in Ψ it can easily be shown that exactly the same result is obtained for Box-Jenkins models (under the same order condition on n_b and n_f). Thus an input with spectral factor $R(z) = 1/(z - z_o^{-1})$ ensures that the asymptotic variance of an estimate of an NMP-zero at z_o becomes independent both of the model and system order and also the model structure. One may argue that this choice of input is infeasible since it depends on the to be estimated zero. However, this insight is of independent value; it has been shown that using an estimate of the NMP-zero in R instead, an adaptive scheme will achieve the optimal asymptotic variance, under quite mild conditions, see Rojas et al. (2011).

7 Conclusions

The main results in this paper are Theorem 1 and Theorem 4 which are the result of careful reparametrization of formulas found in Mårtensson and Hjalmarsson (2011). With this novel reparametrization, these theorems express the asymptotic covariance as defined by (5). We have shown that these geometric expressions provide insights into how model structure, model order, true system dynamics, and experimental conditions affect the asymptotic covariance. In particular, we demonstrated that one can use the experimental conditions to make the asymptotic variance independent of model order and model structure in some cases.

We have also used these expressions to derive novel model structure independent upper bounds of the asymptotic covariance, in particular for a number of commonly estimated quantities such as system zeros and gains and impulse response coefficients. We have shown that these bounds are significantly less conservative as compared to the variance expressions that result from using the (asymptotic in model order) variance formulae for frequency function estimation in Ljung (1985).

Our work has its foundation in Ninness and Hjalmarsson (2004), where the significance of the subspace spanned by the prediction error gradient was acknowledged and we have shown that the results in Ninness and Hjalmarsson (2004) are recovered from the results in this paper.

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